Actuarial fire-spreading model based on tree-structured graphical models, with insurance applications

with O. Chuisseu, JP Boucher, and H Cossette

Etienne Marceau

Professeur titulaire, PhD, ASA
École d'actuariat, Faculté des Sciences et de Génie
Université Laval, Québec, Canada

AFRIC 2 | The 2025 Actuarial, Finance, Risk and Insurance Congress 2
Maasai Mara Edition, Sunday 8 - Friday 13 June 2025
Wednesday 11 June 2025, Session W2.3 12-13pm (Kenya Time)

Université Laval | Faculté des sciences et de génie, École d'actuariat | Quantact | CIMMUL | CIRCERB | crdm.ul | Institut intelligence et données

Foreword

Thank you to . . .

- . . . Jonathan, Mukami, Patrick, and John
- . . . the members of the organizing and scientific committees of AFRIC2
- . . . Anthony and Philippe for technical support
- . . . Christian Dagenais, Pierre Blanchet and Hailey Quiqero for comments
- . . . my co-authors Orelien, Jean-Phillipe and Hélène
- . . . my teammates in LMH

Introduction and motivations

The need to provide adequate protection against fire hazards has contributed to the emergence and development of P&C Insurance.

Adequate fire hazard modeling is crucial for P&C actuaries.

- In [Parodi, 2023], the author provides an overview of P&C insurance pricing.
- In [Wuthrich, 2023], we find an overview of the actuarial mathematics and statistical methods in P&C insurance.
- See [Barrois, 1835] about one of the first probabilistic model about fire hazard in insurance.

Fire hazard modeling is also essential for civil/fire engineers.

- In [Rychlik and Rydén, 2006], we find an introduction to the approach adopted by civil engineers in fire hazard modeling.
- Academics and practionners in civil/fire engineering focus on models for fire-propagation: see, e.g., [Ling and Williamson, 1985], [Cheng and Hadjisophocleous, 2009], [Kim et al., 2016], [Jiang et al., 2025], etc.

Introduction and motivations

P&C Actuaries vs Civil/Fire Engineers:

- As mentioned in [Wuthrich, 2023] and [Albrecher et al., 2017], the actuarial approach to modeling fire loss costs relies on calibrating appropriate models using historical data.
- This approach does not aim to model the spread of fire within property structures in order to study the random behavior of fire loss costs.
- Engineers have considered approaches based on fire-spreading in models: [Ling and Williamson, 1985], [Cheng and Hadjisophocleous, 2009], [Jiang et al., 2025], etc.
- Actuarial academics and practitioners have rarely addressed this aspect of fire loss modeling.
- Two exceptions: [Parodi and Watson, 2019] and [Boucher et al., 2024].
- Civil/fire engineers: purposes of fire modeling = safety, security mananagement, prevention, life hazard $\Rightarrow$ Modeling relies on understanding fire spreading.
- Actuaries: purposes of fire modeling = damage losses, replacement costs of property losses $\Rightarrow$ Modeling heavily relies on historical data.

Modeling based on historical data has advantages, but also disavantages:

- It is problematic for new construction products to be insured = lack of historical data.
- It may not take into account recent advances in construction methods and products.

Objectives

General objective:

- Study the propagation of fire within residential and commercial property structures using graphical models in order to model the losses due to fire hazard.

To achieve this general objective, we have the following specific objectives:

1. **Specific Objective 1:** Introduce an actuarial fire-speading model based on graphical models defined on a tree and analyze its properties.
2. **Specific Objective 2:** Study the distribution of losses due to fire hazard generated from the model introduced in **Specific Objective** 1.
3. **Specific Objective 3:** Calibrate the model components taking into account the characteristics of the structure studied (design of walls, doors, etc.), the implemented passive fire systems, and examine applications of the model.

Objectives

Additional motivation: Compensate the lack of data on innovative construction products (mass timber) in insurance pricing and integrate recent advances obtained by engineers in construction methods and products.

Modeling philosophy: In line with [Parodi et al., 2024], we adopt a first-principles approach, grounding our model in the mechanisms driving fire propagation and loss generation.

- **Physically grounded:** Our model try to reflect the real-world dynamics of fire spread between adjacent units, rather than relying on purely statistical adjustment.
- **Structured dependencies:** Dependencies are explicitly encoded through a tree-based graphical model, capturing spatial adjacency and propagation paths.
- **Reduced arbitrariness:** Like [Parodi et al., 2024], we argue that choosing models based solely on statistical fit especially with flexible parametric forms can obscure key risk drivers.
- **Practical benefits:** In cases where little data is available (e.g., for mass timber), engineering knowledge helps reduce uncertainty and improves model reliability.

Objectives

We aim to achieve Goals 9 and 11 for sustainable development among the 17 goals established by the United Nations (link 17 goals)

- **Goal 9:** Build resilient infrastructure, promote inclusive sustainable industrialization and encourage innovation;
- **Goal 11:** Ensure that cities and human settlements are inclusive, safe, resilient and sustainable.

Agenda:

- Desired properties and ingredients of fire-spreading models
- Graph theory, probabilistic graphical models, and Markov random fields
- Actuarial fire-spreading model
- Impact of propagation probabilities
- Illustration of the impact of propagation probabilities
- Mass timber construction and insurance challenges

Desired properties and ingredients of fire-spreading models

Let the rv X be the aggregate loss amount due to the fire hazard of a building

$$X_\nu = \begin{cases} \sum_{k=1}^{M_\nu} B_k, & M_\nu > 0, \\ 0, & M_\nu = 0, \end{cases}$$

where

- $M_\nu \sim Pois(\nu\lambda)$: number of fire incidents for a given exposure ν;
- exposure ν: floor area of the property in m^2;
- $\lambda =$ fire rate per m^2; example: $\lambda = 7.30 \times 10^{-6}$, [Gaskin and Yung, 1993]

  supermarket with floor area $\nu = 1500\mathrm{m}^2 \Rightarrow \nu\lambda = 0.01095 = E[M_\nu]$;

- B_k : loss amount due the kth fire incident, for $k \in \mathbb{N}_1 = \{1,2,\dots\}$.

Assumptions:

- $\{B_k, k \in \mathbb{N}_1\}$: sequence of iid random variables where $B_k \overset{d}{=} B$ with cdf F_B.
- $\{B_k, k \in \mathbb{N}_1\}$ and M_ν are independent.

Question: How to *find* F_B if the actuary has no data?

Desired properties and ingredients of fire-spreading models

Answer $\Rightarrow$ Objective of fire-spreading models: Find the distribution of loss amount rv B by describing the mechanism of fire ignition and spreading using probabilistic models.

Desired properties of fire-spreading models:

1. Adaptability allowing to integrate recent findings and advances in civil/fire engineering.
2. Flexibility allowing to integrate different types of material.
3. Interpretability and easy to implement.
4. Bridge between fire research made by civil/fire engineers and actuaries.

Key ingredients to fire-spreading models ([Cheng and Hadjisophocleous, 2009]):

1. Characteristics:
   - fuel type, fuel load, compartment geometry and ventilation conditions.
2. Ignition:
   - ignition source, ignition location, fuel arrangement.
3. Types of fires:
   - smouldering fires, non-flashover flaming fires, flashover fires.

Desired properties and ingredients of fire-spreading models

There are few fire-spreading models in the actuarial literature.

Models from actuarial literature:

- [Parodi and Watson, 2019]:
  - Their model is based on random graphs
  - One of main objectives of the authors is to draw a connection between random graphs and Bernegger's curves introduced in [Bernegger, 1997].
  - See also Section 4.2 in [Parodi et al., 2024]
- [Boucher et al., 2024]
  - Contagion-based model
  - Built using insurance data from Canadian farms

We recall notations and definitions from graph theory.

A tree $\mathcal{T}$ is an undirected graph defined by a pair $(\mathcal{V},\mathcal{E})$:

- $\mathcal{V} = \{1,2,\dots,d\}$ = set of vertices
- $\mathcal{E}$ = set of $d-1$ edges
- $\mathrm{path}(u,v)$: path from vertex u to vertex v
  - $\mathrm{path}(u,v)$: sequence of successive edges $e \in \mathcal{E}$
  - the first edge starts at vertex u
  - the last edge ends at vertex v
  - the same edge cannot appear more than once in $\mathrm{path}(u,v)$
- In $\mathcal{T}$: there is no path from a vertex to itself.

$\mathcal{T}_r$: rooted version of a tree $\mathcal{T}$:

- $r \in \mathcal{V}$ = one specific vertex labelled as the *root*
- For an undirected tree $\mathcal{T}$, one can choose any vertex $r \in \mathcal{V}$ to be the root
- descendents of vertex $v = \mathrm{dsc}(v)$, $v \in \mathcal{V}$:
  - set of vertices whose path to the root r goes through v
- children of vertex $v = \mathrm{ch}(v)$, $v \in \mathcal{V}$:
  - set of vertices descendents of v that are also connected to v by an edge
- leaf = vertex that has no children
- parent of vertex $v = \mathrm{pa}(v)$, $v \in \mathcal{V} \setminus \{r\}$:
  - the sole vertex connected by an edge to v that is not the children of v
- The root r has no parent

Working with rooted version $\mathcal{T}_r$ of a tree $\mathcal{T}$ helps to build the fire-spreading model.

Example: Consider a rooted version $\mathcal{T}_1$ of tree $\mathcal{T} = (\mathcal{V},\mathcal{E})$ with

- $\mathcal{V} = \{1,2,3,4,5,6,7\}$
- $\mathcal{E} = \{(1,2),(1,3),(3,4),(3,5),(4,6),(4,7)\}$

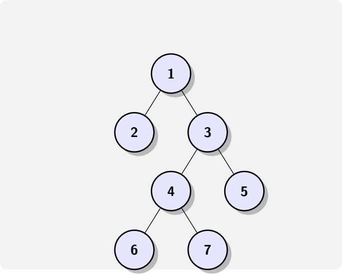


Figure: Example of a binary tree

- **Root**: $\{1\}$
- **Leaves**: $\{2,5,6,7\}$
- **Descendants of 3**: $\mathrm{dsc}(3) = \{4,5,6,7\}$
- **Children of 3**: $\mathrm{ch}(3) = \{4,5\}$; $\mathrm{ch}(2) = \emptyset$
- **Parent of 4**: $\mathrm{pa}(4) = \{3\}$
- **Path from 1 to 7**: $\mathrm{path}(1,7) = \{(1,3),(3,4),(4,7)\}$

Probabilistic graphical models

Probabilistic graphical models (PGMs) provide a unifying framework:

- They capture complex dependence relations among random variables.
- They provide a visual representation of the structure of a multivariate distribution of the vector of random variables from which conditional independence relationships can be inferred between them.
- See, e.g., [Lauritzen, 1996], [Wainwright et al., 2008], [Koller and Friedman, 2009].

Two large families of PGMs:

- Bayesian Networks, also called Directed Graphical Models.
- Markov random fields, also called Undirected Graphical Models.
- Fire-propagation models from civil/fire engineering are based on Bayesian Networks.
- Our approach is based on Tree-Structured Markov random fields.
- Every Bayesian Network can be converted into a MRF, [Koller and Friedman, 2009].

Markov random fields

Markov random fields (MRFs) are used to model high dimensional vector of rvs:

1. Each node in the graph corresponds to a random variable.
2. Each edge between two adjacent nodes (rvs) captures direct interactions, which make the model interpretable.

Examples of classical MRFs: MRFs with Bernoulli marginals (see *Ising model*).

Our fire-spreading model is based on tree-structured MRFs with Bernoulli marginals.

Tree-structured MRFs with Bernoulli marginals:

The rvs are required to satisfy the Markov property, whereby a rv is conditionally independent of all other rvs given its neighbors.

Actuarial fire-spreading model

Definitions

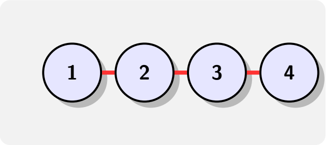
In the actuarial fire-spreading model, the fire-propagation is defined on a tree.

Definitions:

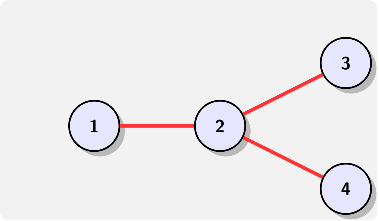
- Site of d units: a building (residential, commercial), a farm, a construction site.
- Unit: an individual room of a building or a part of a construction site.
- Representation of the structure of the site of d units: tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$.
- $\mathcal{V} = \{1, \ldots, d\}$: set of vertices representing the d units.
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$: set of edges.
- $(v_1, v_2) \in \mathcal{E}$: edge (link) between connected unit v_1 and unit v_2.

Actuarial fire-spreading model

Definitions



(a) Building with 4 units in row, represented by $\mathcal{T}^{4,1}$ (serie tree)



(b) Building with 4 units in hub, represented by $\mathcal{T}^{4,2}$ (hub tree)

Figure: Two tree structures representing a 4-unit building

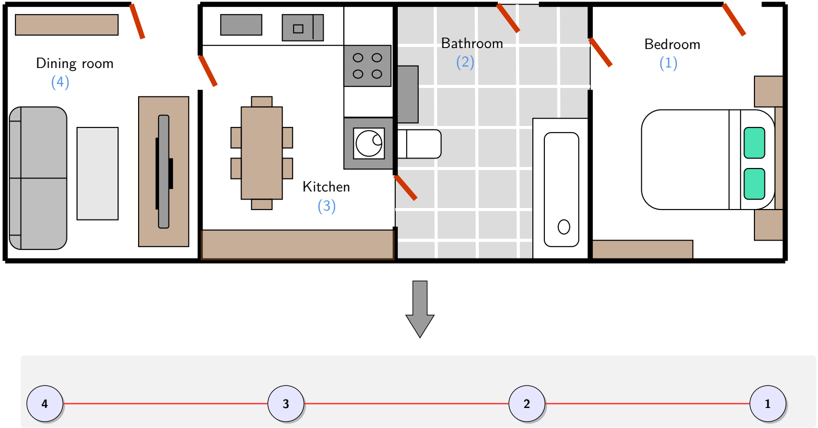
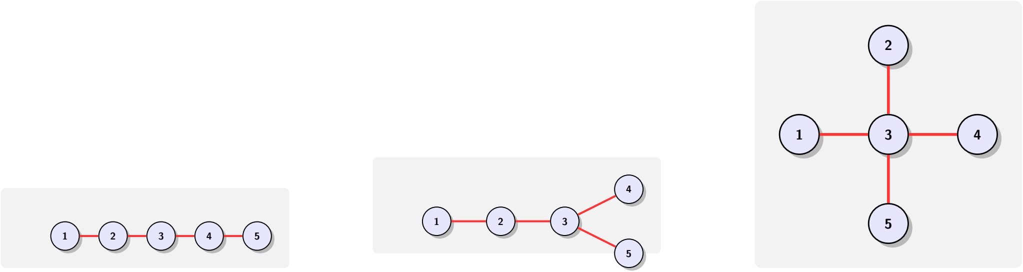


Figure: Building with 4 units in a row, represented by the series tree $\mathcal{T}^{4,1}$. For other similar plans, see Figure 2 in [Li et al., 2013].

Actuarial fire-spreading model

Definitions



(a) Building with 5 units in row, represented by $\mathcal{T}^{5,1}$ (series tree)

(b) Building with 5 units in hub, represented by $\mathcal{T}^{5,2}$ (hub tree)

(c) Building with 5 units in hub, represented by $\mathcal{T}^{5,3}$ (hub tree)

Figure: Three tree structures representing a 5-unit building

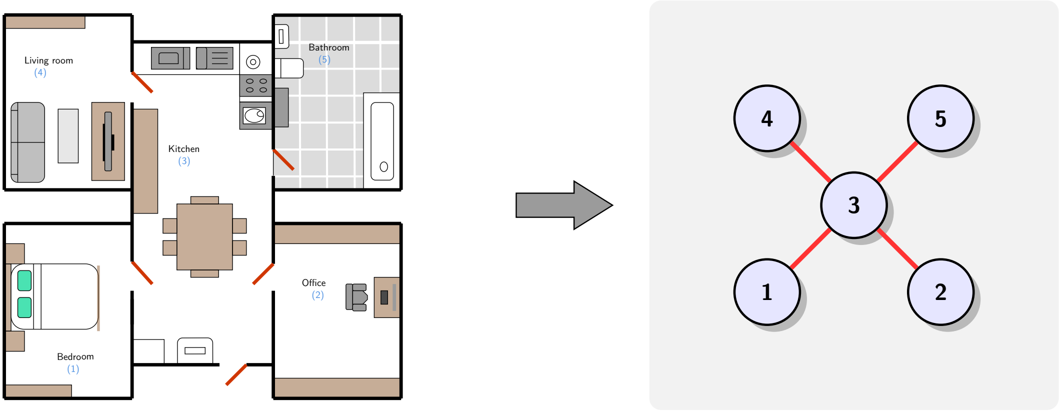


Figure: Building with 5 units in hub, represented by hub (or star) tree $\mathcal{T}^{5,3}$. For other similar plans, see Figure 2 in [Li et al., 2013]

Actuarial fire-spreading model

Definitions

Objective of the actuarial fire-spreading model: Derive/find the distribution of loss amount rv B by describing the mechanism of fire ignition and spreading using a probabilistic graphical model.

Assume that a fire occurs on a site of d units with a given structure $\mathcal{T} = (\mathcal{V},\mathcal{E})$:

- $B = \sum_{v\in\mathcal{V}} D_v$ = loss amount rv = sum of eventual damages to the d units;
- $D_v = \begin{cases} Y_v, & I_v = 1 \\ 0, & I_v = 0 \end{cases}$ = eventual damages rv due to fire at unit $v \in \mathcal{V}$;
- I_v = Bernoulli rv = $\begin{cases} 1, & \text{if the unit } v \text{ burns down} \\ 0, & \text{otherwise} \end{cases}$, for $v \in \mathcal{V}$;
- $Y_v \in [0,c_v]$ = amount of damage rv in unit $v \in \mathcal{V}$;
- c_v = property value (or insured value) of unit $v \in \mathcal{V}$.

Remarks:

- The distribution of the loss amount rv B depends on the multivariate distribution of the vector of d Bernoulli rvs $\boldsymbol{I} = (I_v, v \in \mathcal{V})$.
- The multivariate distribution of $\boldsymbol{I}$ relies on the structure $\mathcal{T} = (\mathcal{V},\mathcal{E})$.

Focus:

- Derive the multivariate distribution of $\boldsymbol{I}$ using an approach based on tree-structured MRFs with Bernoulli marginals.
- Find the distribution of B from the multivariate distribution of $\boldsymbol{I}$.
- Investigate the distribution of B and applications.

Actuarial fire-spreading model

Fire ignition location rv

The multivariate distribution of $\boldsymbol{I}$ is characterized by its pmf:

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \Pr(I_1 = i_1, \ldots, I_d = i_d), \quad \boldsymbol{i} = (i_1, \ldots, i_d) \in \{0,1\}^d.$$

We compute the values of $f_{\boldsymbol{I}}$ using the following 4 components:

- Fire ignition location.
- Fire flashover probabilities at fire ignition locations.
- Configuration of the tree structure $\mathcal{T}$ of the building.
- Propagation probabilities.

We define the discrete rv Θ as the fire ignition location rv:

- Pmf f_Θ of Θ: $f_\Theta(s) = \Pr(\Theta = s) \geq 0, \quad$ for $s \in \mathcal{V}$.
- $\{\Theta = s\}$ means "the fire ignition location is unit $s \in \mathcal{V}$".
- See Figures 4.14-4.21 in [Mou, 2009] for examples of values of the pmf f_Θ of the fire ignition location rv for a building of $d = 5$ units. See also [Li et al., 2013].

Actuarial fire-spreading model

Fire ignition location rv

Let the fire ignition location be unit s, i.e. $\Theta = s$.

At the fire ignition location, the fire may flashover or not:

- Fire Flashover Probability: The fire that started within unit s sets fire to the unit s, with probability $p_s \in (0,1)$:

$$\Pr(I_s = 1 | \Theta = s) = p_s$$

- No Fire Flashover Probability: The fire dies at its source and leaves the unit s intact with probability $q_s = 1 - p_s$.

$$\Pr(I_s = 0 | \Theta = s) = q_s = 1 - p_s$$

When a fire occurs on a site, it results from only one fire ignition location.

Actuarial fire-spreading model

Propagation probabilities

Fire propagation between units is governed by the tree structure $\mathcal{T} = (\mathcal{T}, \mathcal{E})$:

- Each edge $(v_1, v_2) \in \mathcal{E}$ is associated with a propagation probability p_{v_1,v_2}.
- A fire can only spread either from unit v_1 to connected unit v_2 or from unit v_2 to connected unit v_1, for edge $(v_1, v_2) \in \mathcal{E}$.

We introduce the very small arrow $\scriptstyle\rightarrow$ to indicate the direction of the fire-spreading:

- $p_{v_1 \rightarrow v_2} = p_{v_1,v_2} =$ probability that the fire spreads from unit v_1 to connected unit v_2.
- $p_{v_2 \rightarrow v_1} = p_{v_1,v_2} =$ probability that the fire spreads from unit v_2 to connected unit v_1.
- $q_{v_1 \rightarrow v_2} = 1 - p_{v_1,v_2} =$ probability that the fire does not spread from unit v_1 to connected unit v_2.
- $q_{v_2 \rightarrow v_1} = 1 - p_{v_1,v_2} =$ probability that the fire does not spread from unit v_2 to connected unit v_1.

Propagation probabilities = crucial components:

- p_{v_1,v_2} are the crucial components of the actuarial fire spreading model.
- p_{v_1,v_2} depend on the effectiveness of fire barriers between units, etc.

The joint pmf of $\boldsymbol{I}$ is given by

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \sum_{s \in \mathcal{V}} f_{\Theta}(s) \times f_{\boldsymbol{I}|\Theta=s}(\boldsymbol{i}), \quad \text{for } \boldsymbol{i} \in \{0,1\}^d. \tag{1}$$

We find the conditional pmf $f_{\boldsymbol{I}|\Theta=s}$ as follows:

- Given that a fire-ignition location is s, i.e. $\{\Theta = s\}$, choose s to be the root.
- Let $\mathcal{T}_s$ be the rooted version of $\mathcal{T}$.
- Induced directness: The fire spreads from the root s through its descendants.
- Application of global Markov property for tree-structured MRF assuming $\mathcal{T}_s$:

  The probability of fire-spreading to unit v depends only on the status of unit v's parent $\mathrm{pa}(v)$ (burned-down or not) and not on the fire trajectory before hitting that specific unit $\mathrm{pa}(v)$.

Based on the rooted version $\mathcal{T}_s$ of $\mathcal{T}$, we find the values of

$$\Pr(I_v = i_v | I_{\mathrm{pa}(v)} = i_{\mathrm{pa}(v)}), \ \forall v \in \mathrm{dsc}(s), \tag{2}$$

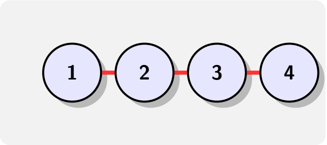
provided in the following table:

$i_{\mathrm{pa}(v)} \nrightarrow i_v$	0	1
0	1	0
1	$1 - p_{\mathrm{pa}(v) \nrightarrow v}$	$p_{\mathrm{pa}(v) \nrightarrow v}$

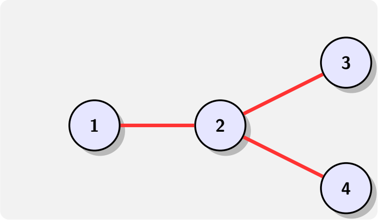
For each $s \in \mathcal{V}$, we compute the 2^d values of the joint conditional pmf of $(\boldsymbol{I} | \Theta = s)$ from

$$f_{\boldsymbol{I}|\Theta=s}(\boldsymbol{i}) = \Pr(I_s = i_s | \Theta = s) \prod_{v \in \mathrm{dsc}(s)} \Pr(I_v = i_v | I_{\mathrm{pa}(v)} = i_{\mathrm{pa}(v)}), \ \boldsymbol{i} \in \{0,1\}^d. \tag{3}$$

Example: In the next two slides, we provide the 2^d values of pmf $f_{\boldsymbol{I}|\Theta=s}$ in (3) for the two possible structures of $d = 4$ units.



(a) Building with 4 units in row, represented by $\mathcal{T}^{4,1}$ (serie tree)



(b) Building with 4 units in hub, represented by $\mathcal{T}^{4,2}$ (hub tree)

Actuarial fire-spreading model

Conditional joint pmf of the Bernoulli random vectors

i_1	i_2	i_3	i_4	$f_{\boldsymbol{I}\mid\Theta=1}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=2}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=3}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=4}(\boldsymbol{i})$
1	0	0	0	$p_1 q_{1\to2}$	0	0	0
1	0	0	1	0	0	0	0
1	0	1	0	0	0	0	0
1	0	1	1	0	0	0	0
1	1	0	0	$p_1 p_{1\to2} q_{2\to3}$	$p_2 p_{2\to1} q_{2\to3}$	0	0
1	1	0	1	0	0	0	0
1	1	1	0	$p_1 p_{1\to2} p_{2\to3} q_{3\to4}$	$p_2 p_{2\to3} p_{2\to1} q_{3\to4}$	$p_3 p_{3\to2} p_{2\to1} q_{3\to4}$	0
1	1	1	1	$p_1 p_{1\to2} p_{2\to3} p_{3\to4}$	$p_2 p_{2\to3} p_{2\to1} p_{3\to4}$	$p_3 p_{3\to2} p_{2\to1} p_{3\to4}$	$p_4 p_{4\to3} p_{3\to2} p_{2\to1}$
0	0	0	0	q_1	q_2	q_3	q_4
0	0	0	1	0	0	0	$p_4 q_{4\to3}$
0	0	1	0	0	0	$p_3 q_{3\to2} q_{3\to4}$	0
0	0	1	1	0	0	$p_3 q_{3\to2} p_{3\to4}$	$p_4 p_{4\to3} q_{3\to2}$
0	1	0	0	0	$p_2 q_{2\to3} q_{2\to1}$	0	0
0	1	0	1	0	0	0	0
0	1	1	0	0	$p_2 p_{2\to3} q_{2\to1} q_{3\to4}$	$p_3 p_{3\to2} q_{2\to1} q_{3\to4}$	0
0	1	1	1	0	$p_2 p_{2\to3} q_{2\to1} p_{3\to4}$	$p_3 p_{3\to2} q_{2\to1} p_{3\to4}$	$p_4 p_{4\to3} p_{3\to2} q_{2\to1}$

Table: $f_{\boldsymbol{I}\mid\Theta=s}$, for $s \in \mathcal{V}$, for the tree $\mathcal{T}^{4,1}$

i_1	i_2	i_3	i_4	$f_{\boldsymbol{I}\mid\Theta=1}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=2}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=3}(\boldsymbol{i})$	$f_{\boldsymbol{I}\mid\Theta=4}(\boldsymbol{i})$
1	0	0	0	$p_1 q_{1\to 2}$	0	0	0
1	0	0	1	0	0	0	0
1	0	1	0	0	0	0	0
1	0	1	1	0	0	0	0
1	1	0	0	$p_1 p_{1\to 2} q_{2\to 3} q_{2\to 4}$	$p_2 p_{2\to 1} q_{2\to 3} q_{2\to 4}$	0	0
1	1	0	1	$p_1 p_{1\to 2} q_{2\to 3} p_{2\to 4}$	$p_2 p_{2\to 1} q_{2\to 3} p_{2\to 4}$	0	$p_4 p_{4\to 2} p_{2\to 1} q_{2\to 3}$
1	1	1	0	$p_1 p_{1\to 2} p_{2\to 3} q_{2\to 4}$	$p_2 p_{2\to 1} p_{2\to 3} q_{2\to 4}$	$p_3 p_{3\to 2} p_{2\to 1} q_{2\to 4}$	0
1	1	1	1	$p_1 p_{1\to 2} p_{2\to 3} p_{2\to 4}$	$p_2 p_{2\to 1} p_{2\to 3} p_{2\to 4}$	$p_3 p_{3\to 2} p_{2\to 1} p_{2\to 4}$	$p_4 p_{4\to 2} p_{2\to 1} p_{2\to 3}$
0	0	0	0	q_1	q_2	q_3	q_4
0	0	0	1	0	0	0	$p_4 q_{4\to 2}$
0	0	1	0	0	0	$p_3 q_{3\to 2}$	0
0	0	1	1	0	0	0	0
0	1	0	0	0	$p_2 q_{2\to 1} q_{2\to 3} q_{2\to 4}$	0	0
0	1	0	1	0	$p_2 q_{2\to 1} q_{2\to 3} p_{2\to 4}$	0	$p_4 p_{4\to 2} q_{2\to 1} q_{2\to 3}$
0	1	1	0	0	$p_2 q_{2\to 1} p_{2\to 3} q_{2\to 4}$	$p_3 p_{3\to 2} q_{2\to 1} q_{2\to 4}$	0
0	1	1	1	0	$p_2 q_{2\to 1} p_{2\to 3} p_{2\to 4}$	$p_3 p_{3\to 2} q_{2\to 1} p_{2\to 4}$	$p_4 p_{4\to 2} q_{2\to 1} p_{2\to 3}$

Table: $f_{\boldsymbol{I}\mid\Theta=s}$, for $s \in \mathcal{V}$, for the tree $\mathcal{T}^{4,2}$

Actuarial fire-spreading model

Joint pmf of the Bernoulli random vectors

Summary:

- $(\boldsymbol{I}|\Theta = s)$ is a MRF with Bernoulli marginals and its joint pmf is given by (3).
- Distribution of $\boldsymbol{I}$: finite mixture of multivariate Bernoulli distributions associated with MRFs $(\boldsymbol{I}|\Theta = 1)$, ..., $(\boldsymbol{I}|\Theta = d)$, and its joint pmf given by

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \sum_{s \in \mathcal{V}} f_{\Theta}(s) \times f_{\boldsymbol{I}|\Theta=s}(\boldsymbol{i}), \quad \text{for } \boldsymbol{i} \in \{0,1\}^d.$$

- The multivariate Bernoulli distribution of $\boldsymbol{I}$ is characterized by 4 components:
  - Distribution of the fire ignition location rv Θ;
  - Fire flashover probabilities at fire ignition locations;
  - Structure $\mathcal{T} = (\mathcal{V},\mathcal{E})$ of the site;
  - Propagation probabilities $\boldsymbol{p} = (p_{v_1,v_2}, (v_1,v_2) \in \mathcal{E})$ on the edges of $\mathcal{T}$.

Let $b = \sum_{v \in \mathcal{V}} c_v$ be the total property value (or total insured value) of the site.

The cdf of the loss amount rv B is given by

$$F_B(x) = f_{\boldsymbol{I}}(\boldsymbol{0}) + \sum_{\boldsymbol{i} \in \{0,1\}^d \setminus \{\boldsymbol{0}\}} f_{\boldsymbol{I}}(\boldsymbol{i}) F_{i_1 Y_1 + \cdots + i_d Y_d}(x), \quad x \in [0,b]. \tag{4}$$

The cdf of B is also given by

$$F_B(x) = \sum_{s \in \mathcal{V}} f_{\Theta}(s) F_{B|\Theta = s}(x), \quad x \in [0,b].$$

Note: $F_B(x) = 1$ for $x > b$.

Actuarial fire-spreading model

Loss amount rv B

Building 4-units in row $\mathcal{T}^{4,1}$: The conditional cdf of $(B^{\mathcal{T}^{4,1}}|\Theta = s)$, $x \in [0,b]$, is

$$F_{B^{\mathcal{T}^{4,1}}|\Theta=s}(x) = \begin{cases} q_1 + p_1\Big(q_{1\to 2}F_{Y_1}(x) + p_{1\to 2}q_{2\to 3}F_{Y_1+Y_2}(x) \\ \quad +p_{1\to 2}p_{2\to 3}q_{3\to 4}F_{Y_1+Y_2+Y_3}(x) + p_{1\to 2}p_{2\to 3}p_{3\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x)\Big), & s=1, \\ q_2 + p_2\Big(q_{2\to 1}q_{2\to 3}F_{Y_2}(x) + p_{2\to 1}q_{2\to 3}F_{Y_1+Y_2}(x) \\ \quad +q_{2\to 1}p_{2\to 3}q_{3\to 4}F_{Y_2+Y_3}(x) + p_{2\to 1}p_{2\to 3}q_{3\to 4}F_{Y_1+Y_2+Y_3}(x) \\ \quad +q_{2\to 1}p_{2\to 3}p_{3\to 4}F_{Y_2+Y_3+Y_4}(x) + p_{2\to 1}p_{2\to 3}p_{3\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x)\Big), & s=2, \\ q_3 + p_3\Big(q_{3\to 2}q_{3\to 4}F_{Y_3}(x) + p_{3\to 2}q_{2\to 1}q_{3\to 4}F_{Y_2+Y_3}(x) \\ \quad +q_{3\to 2}p_{3\to 4}F_{Y_3+Y_4}(x) + p_{3\to 2}p_{2\to 1}q_{3\to 4}F_{Y_1+Y_2+Y_3}(x) \\ \quad +p_{3\to 2}q_{2\to 1}p_{3\to 4}F_{Y_2+Y_3+Y_4}(x) + p_{3\to 2}p_{2\to 1}p_{3\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x)\Big), & s=3, \\ q_4 + p_4\Big(q_{4\to 3}F_{Y_4}(x) + p_{4\to 3}q_{3\to 2}F_{Y_3+Y_4}(x) \\ \quad +p_{4\to 3}p_{3\to 2}q_{2\to 1}F_{Y_2+Y_3+Y_4}(x) + p_{4\to 3}p_{3\to 2}p_{2\to 1}F_{Y_1+Y_2+Y_3+Y_4}(x)\Big), & s=4. \end{cases}$$

Building 4-units in hub $\mathcal{T}^{4,2}$: The conditional cdf of $(B^{\mathcal{T}^{4,2}}|\Theta = s)$, $x \in [0,b]$, is

$$F_{B^{\mathcal{T}^{4,2}}|\Theta=s}(x) = \begin{cases} q_1 + p_1\Big(q_{12}F_{Y_1}(x) + p_{1\to 2}q_{2\to 3}q_{2\to 4}F_{Y_1+Y_2}(x) + p_{1\to 2}q_{2\to 3}p_{2\to 4}F_{Y_1+Y_2+Y_4}(x) \\ \quad +p_{1\to 2}p_{2\to 3}q_{2\to 4}F_{Y_1+Y_2+Y_3}(x) + p_{1\to 2}p_{2\to 3}p_{2\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x)\Big), & s=1, \\ q_2 + p_2\Big(p_{2\to 1}q_{2\to 3}q_{2\to 4}F_{Y_1+Y_2}(x) + p_{2\to 1}q_{2\to 3}p_{2\to 4}F_{Y_1+Y_2+Y_4}(x) + p_{2\to 1}p_{2\to 3}q_{2\to 4}F_{Y_1+Y_2+Y_3}(x) \\ \quad +p_{2\to 1}p_{2\to 3}p_{2\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x) + q_{2\to 1}q_{2\to 3}q_{2\to 4}F_{Y_2}(x) \\ \quad +q_{2\to 1}q_{2\to 3}p_{2\to 4}F_{Y_2+Y_4}(x) + q_{2\to 1}p_{2\to 3}q_{2\to 4}F_{Y_2+Y_3}(x) + q_{2\to 1}p_{2\to 3}p_{2\to 4}F_{Y_2+Y_3+Y_4}(x)\Big), & s=2, \\ q_3 + p_3\Big(p_{3\to 2}q_{2\to 4}p_{2\to 1}F_{Y_1+Y_2+Y_3}(x) \\ \quad +p_{3\to 2}p_{2\to 1}p_{2\to 4}F_{Y_1+Y_2+Y_3+Y_4}(x) + q_{3\to 2}F_{Y_3}(x) \\ \quad +p_{3\to 2}q_{2\to 1}q_{2\to 4}F_{Y_2+Y_3}(x) + p_{3\to 2}q_{2\to 1}p_{2\to 4}F_{Y_2+Y_3+Y_4}(x)\Big), & s=3, \\ q_4 + p_4\Big(p_{4\to 2}p_{2\to 1}q_{2\to 3}F_{Y_1+Y_2+Y_4}(x) + p_{4\to 2}p_{2\to 1}p_{2\to 3}F_{Y_1+Y_2+Y_3+Y_4}(x) \\ \quad +q_{4\to 2}F_{Y_4}(x) + p_{4\to 2}q_{2\to 1}q_{2\to 3}F_{Y_2+Y_4}(x) + p_{4\to 2}q_{2\to 1}p_{3\to 2}F_{Y_2+Y_3+Y_4}(x)\Big), & s=4. \end{cases}$$

Impact of propagation probabilities

The propagation probabilities $\boldsymbol{p} = (p_{v_1,v_2}, (v_1,v_2) \in \mathcal{E})$ are the key parameters of the actuarial fire-spreading model.

We use the usual stochastic order to examine how an increase/decrease of the propagation probabilities impacts the distribution of the loss amount rv B.

Definition – Usual stochastic order *Let X and X' be rvs with cdf F_X and $F_{X'}$. We say that X is smaller than X' under the usual stochastic order, noted $X \preceq_{st} X'$, if $F_X(x) \geq F_{X'}(x)$, for all $x \in \mathbb{R}$.*

Examples of implications:

- Expectation: If $X \preceq_{st} X'$, then $E[X] \leq E[X']$.
- Value-at-Risk: If $X \preceq_{st} X'$, then $VaR_\kappa(X) \leq VaR_\kappa(X')$, for $\kappa \in (0,1)$.

Theorem 1 – Impact of the propagation probabilities

Consider two stuctures of d units represented by trees $\mathcal{T} = (\mathcal{V},\mathcal{E})$ and $\mathcal{T}' = (\mathcal{V}',\mathcal{E}')$ with propagation probabilities $\boldsymbol{p}$ and $\boldsymbol{p}'$ such that $\mathcal{V} = \mathcal{V}'$ and $\mathcal{E} = \mathcal{E}'$.

Let B and B' be the loss amount rvs associated to structures $\mathcal{T}$ and $\mathcal{T}'$.

If $0 < p_{v_1,v_2} \leq p'_{v_1,v_2} < 1$, for all $(v_1,v_2) \in \mathcal{E}$, and $Y_v \preceq_{st} Y'_v$, for all $v \in \mathcal{V}$, then

- $(B|\Theta = s) \preceq_{st} (B'|\Theta = s)$, for all fire-ignition location $s \in \mathcal{V}$;
- $B \preceq_{st} B'$.

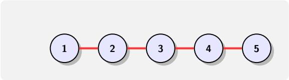
Theorem 1 is crucial for applications in actuarial science and civil/fire engineering:

- Reinforcing fire safety measures on the site, such as adding fire doors or sprinklers, reduces the probabilities of fire propagation.
- By Theorem 1, reducing the probabilities of fire propagation decreases the magnitude of the losses due to fire.
- With Theorem 1, we directly link the introduction of fire prevention systems to risk reduction and insurance premium reduction.

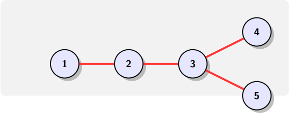
Illustration of the impact of propagation probabilities

Objective:

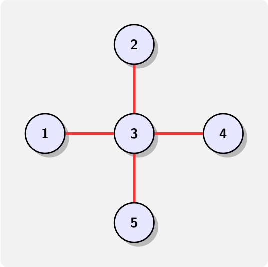
- To illustrate the implications of **Theorem 1** within the three possible structures of 5 units.
- Fire prevention: impact of introducing passive fire protection systems between the connected units (ex: fire door).



(a) Building with 5 units in row, represented by $\mathcal{T}^{5,1}$ (series tree)

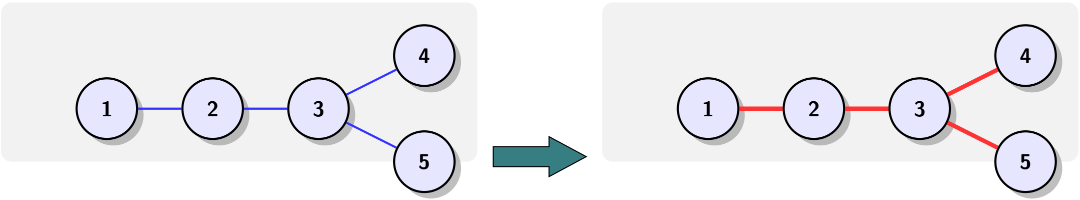


(b) Building with 5 units in hub, represented by $\mathcal{T}^{5,2}$ (hub tree)



(c) Building with 5 units in hub, represented by $\mathcal{T}^{5,3}$ (hub tree)

Illustration of the impact of propagation probabilities



High Initial Propagation Probabilities
(Without fire doors between units)

Decreased Propagation Probabilities
(With fire doors between units)

Illustration of the impact of propagation probabilities

Assumptions of the model for the trees $\mathcal{T}^{5,1}$, $\mathcal{T}^{5,2}$ and $\mathcal{T}^{5,3}$:

- Propagation probabilities: $p_{v_1,v_2} = p$, for $(v_1,v_2) \in \mathcal{E}^{5,m}$, $m = 1,2,3$.
- Amount of damage rv: $f_{Y_s}(k) = \left(\frac{k}{c_s}\right)^{\alpha_s} - \left(\frac{k-1}{c_s}\right)^{\alpha_s}$, $\quad k \in \{1,\ldots,c_s\}$, $s \in \mathcal{V}$ (discretized beta distribution)
- Table of parameters:

Unit s	Description	α_s	c_s	$f_\Theta(s)$	$\Pr(I_s = 1 \vert \Theta = s)$
1	Bedroom	2	100	0.1	0.2
2	Office	1.5	80	0.1	0.2
3	Kitchen	3	150	0.55	0.4
4	Living room	2.5	120	0.2	0.2
5	Bathroom	1	50	0.05	0.1

$\Rightarrow$ Discrete loss amount rv $B \in \{1,\ldots,b\}$, with $b = c_1 + \cdots + c_5 = 500$.

Comments:

- We fix those assumptions to demonstrate the ability to compute all desired values.
- Values of pmf f_Θ are inspired from those of Section 3.2 in [Li et al., 2013].
- Parameters will be fixed using experts' opinion, such as civil/fire engineers, underwriters, etc.

Illustration of the impact of propagation probabilities

Values of $F_{B\mathcal{T}^{5,1}}(x)$, for $0 \leq x \leq 500$, where $F_{B\mathcal{T}^{5,1}}(0) = 0.695$:

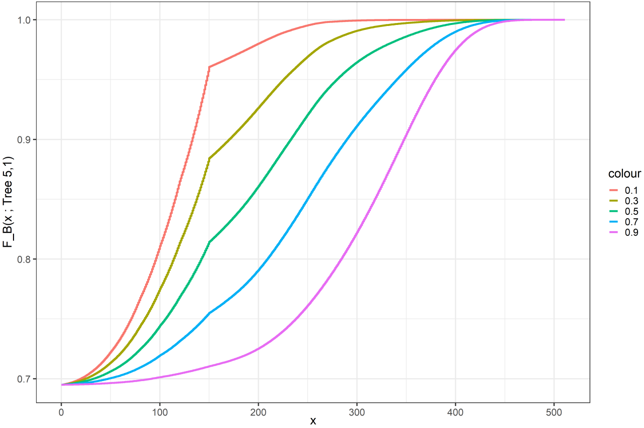


Figure: $F_{B\mathcal{T}^{5,1}}(x)$ for $p = 0.1, 0.3, 0.5, 0.7, 0.9$

Illustration of the impact of propagation probabilities

Values of $F_{B\mathcal{T}^{5,2}}(x)$, for $0 \leq x \leq 500$, where $F_{B\mathcal{T}^{5,2}}(0) = 0.695$:

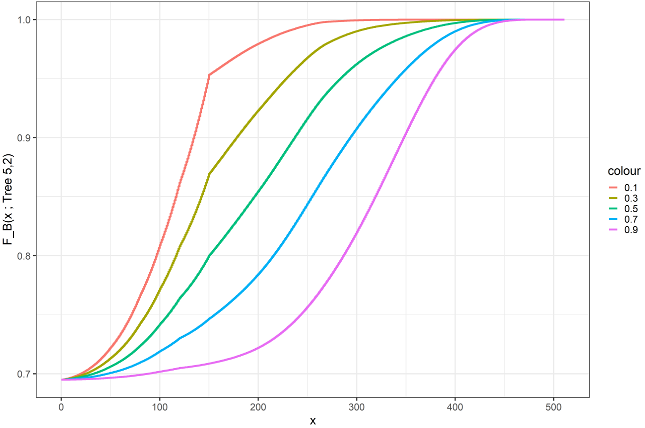


Figure: $F_{B\mathcal{T}^{5,2}}(x)$ for $p = 0.1, 0.3, 0.5, 0.7, 0.9$

Illustration of the impact of propagation probabilities

Values of $F_{B\mathcal{T}^{5,3}}(x)$, for $0 \leq x \leq 500$, where $F_{B\mathcal{T}^{5,3}}(0) = 0.695$:

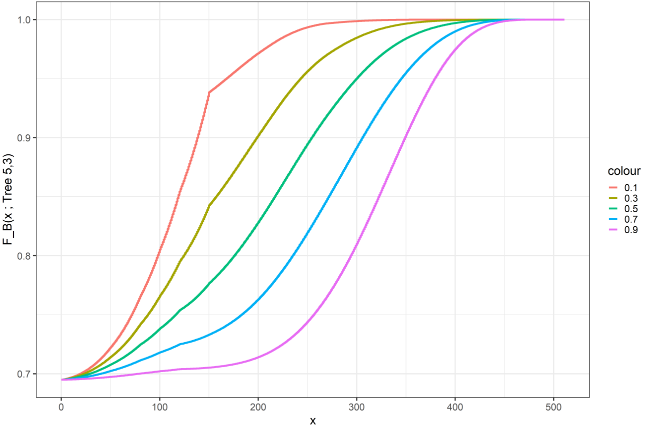


Figure: $F_{B\mathcal{T}^{5,3}}(x)$ for $p = 0.1, 0.3, 0.5, 0.7, 0.9$

Illustration of the impact of propagation probabilities

Values of the expectation and the VaR of the claim amount rvs $B^{\mathcal{T}^{5,1}}, B^{\mathcal{T}^{5,2}}, B^{\mathcal{T}^{5,3}}$:

p	$E[B^{\mathcal{T}^{5,1}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,1}})$	$E[B^{\mathcal{T}^{5,2}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,2}})$	$E[B^{\mathcal{T}^{5,3}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,3}})$
0.9	93	348	93.55	348	95.3	349
0.7	74.02	290	75.21	294	79.07	306
0.5	58.26	233	59.58	237	63.92	255
0.3	45.37	171	46.42	178	49.85	199
0.1	35.04	133	35.47	135	36.85	139

Computational comments:

- All values are exact.
- We compute the values using probability generating functions and the FFT algorithm.

Observations:

- Expectation and VaR of the claim amount rvs $B^{\mathcal{T}^{5,1}}, B^{\mathcal{T}^{5,2}}, B^{\mathcal{T}^{5,3}}$ decrease as propagation probabilities decrease.
- These observations are confirmed by Theorem 1.

MBBEFD: Maxwell–Boltzmann–Bose–Einstein–Fermi–Dirac Distribution

Introduction

- **What is MBBEFD?** A flexible parametric family of distributions inspired by statistical mechanics, used to model losses as proportions of the maximum possible loss.
- **Motivation:** Capture various loss shapes with few parameters. Especially useful in property insurance where loss shapes depend on construction types.
- **Origin:** Introduced by [Bernegger, 1997] for modeling property damage.
- **Connection with graph models:** In [Parodi and Watson, 2019], the authors use (random) graph theory to represent fire and explosion propagation across industrial sites, and proposes that MBBEFD curves can flexibly fit the resulting loss distributions depending on graph structure and ignition points.
- **Applications in reinsurance:** Commonly used to derive exposure curves and severity distributions for reinsurance pricing particularly in fire, industrial, and catastrophe insurance.

Let $Z \in [0,1]$ be a normalized loss (as proportion of MPL). The cdf of the MBBEFD distribution is given by [Parodi and Watson, 2019]:

$$F_Z(x) = \begin{cases} \dfrac{b(g-1)(1-b^x)}{b(g-1)+(1-bg)b^x} & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Parameters: $b > 0$; $g > 1$

Continuity and Regularity:

- $F_Z(x)$ is continuous on $[0,1]$.
- The distribution is strictly increasing on $(0,1)$ under typical parameter values.
- The density function $f_Z(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_Z(x)$ exists and can be evaluated numerically.

In our context:

We fit a parametric MBBEFD distribution to the numerical values of the cdf $F_Z(x)$, which is analytically derived from our actuarial fire-spreading model, in order to approximate the loss distribution using only two interpretable parameters.

Context:

- The loss variable B, derived from our actuarial fire-spreading model, exhibits a significant probability mass at 0.
- This corresponds to cases where no damage occurs (no fire).
- However, the classical MBBEFD distribution is continuous on $[0,1]$ and assigns no probability to zero.

Solution:

- Extend the model to explicitly include a point mass at 0 by defining a two-part (mixed) model.
- This enables us to capture both the frequency of no-loss events and the severity of positive losses.

We define a new random variable W as:

$$W = \begin{cases} Z, & \text{if } K = 1, \\ 0, & \text{if } K = 0, \end{cases}$$

Where:

- $K \sim \text{Bernoulli}(\delta)$ controls the occurrence of damage.
- $Z \sim \text{MBBEFD}(b, g)$ models the loss given that a damage occurred.
- $1 - \delta = F_B(0)$ is estimated from the model-based cdf of B.

The CDF of W is defined as:

$$F_W(x) = \begin{cases} 1-\delta, & \text{if } x = 0, \\ 1-\delta+\delta \times F_Z(\frac{x}{c}), & \text{if } 0 < x \leq c. \end{cases}$$

Interpretation:

- $1-\delta$ is the probability of zero loss.
- F_Z is the cdf of the MBBEFD distribution, which models positive loss values.
- The overall distribution is a mixture of a discrete point at 0 and a continuous distribution on $(0{,}1]$.

Since the rv B is mixed, its cdf admits the following representation:

$$F_B(x) = \begin{cases} 1-\delta, & \text{if } x = 0, \\ 1-\delta+\delta \times F_{B'}(x), & \text{if } 0 < x \leq c, \end{cases}$$

where $\delta = F_B(0)$ and $F_{B'}(x) = F_{B|B>0}(x)$, for $0 < x \leq c$.

We consider two estimation strategies for the MBBEFD parameters (b, g) conditional on a positive loss like presented in [Parodi and Watson, 2019]:

1. **Maximum Likelihood Estimation (MLE)**
   We computed the analytical cumulative distribution function (CDF) of normalized losses and evaluated it at discrete points between 0 and 500. The MBBEFD parameters (b, g) were then estimated by maximizing the likelihood associated with these discretized values.

2. **Least Squares Estimation (LSE)**
   Minimize the squared error between the empirical CDF F_B and the theoretical CDF F_W from the mixed model.

Motivation: The original parameters (b, g) of the MBBEFD distribution can span several orders of magnitude, making optimization unstable see.

Reparameterization:

$$b = e^k, \quad g = 1 + e^\ell$$

Advantages:

- Typical values of k and ℓ fall within more manageable ranges:

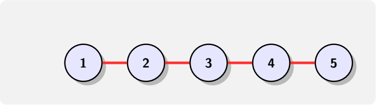
$$k \in [-15, 5], \quad \ell \in [-5, 20]$$

- Improves numerical stability during estimation (MLE or LSE).
- Facilitates convergence in optimization algorithms.

This reparameterization is particularly useful when fitting the MBBEFD to empirical or analytically derived loss distributions, as suggested in [Parodi and Watson, 2019].

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,1}$, $p = 0.3$

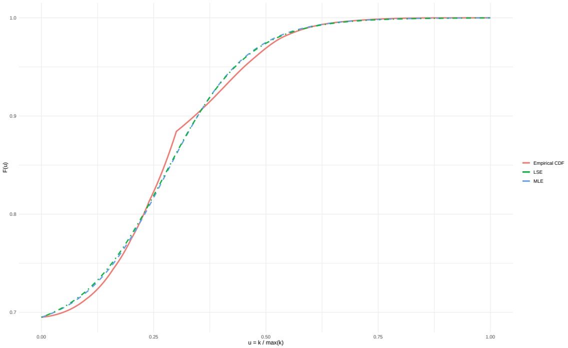
Structure $\mathcal{T}^{5,1}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-11.03	7.93	1.62×10^{-5}	2779.30
LSE	-10.76	7.76	2.13×10^{-5}	2342.52

Table: Estimated parameters on $\mathcal{T}^{5,1}$, with $p = 0.3$

CDF vs MBBEFD Fit

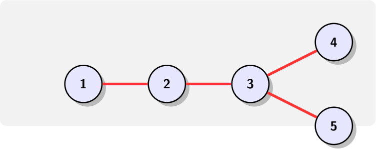


Observations: In $[0, 0.6]$, the MBBEFD model alternates between over- and underestimation. Beyond that, it aligns well with the analytical CDF. This behavior is consistent across both estimation methods.

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,2}$, $p = 0.3$

UNIVERSITÉ LAVAL

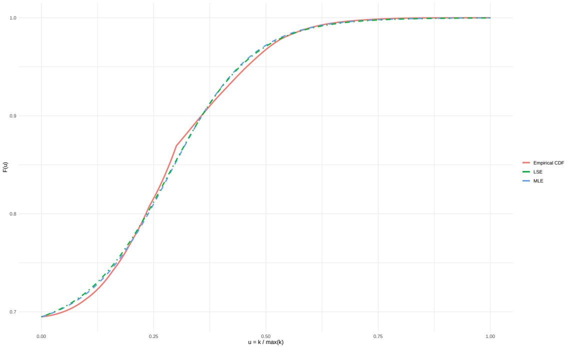
Structure $\mathcal{T}^{5,2}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-10.91	7.75	1.82×10^{-5}	2328.31
LSE	-10.61	7.56	2.47×10^{-5}	1920.57

Table: Estimated Parameters on $\mathcal{T}^{5,2}$, with $p = 0.3$

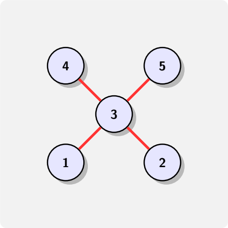
CDF vs MBBEFD Fit



Observations: Similar to the previous case, the model alternates between over- and underestimation in $[0, 0.6]$, then aligns closely with the analytical CDF. This pattern is observed for both estimation methods.

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,3}$, $p = 0.1$

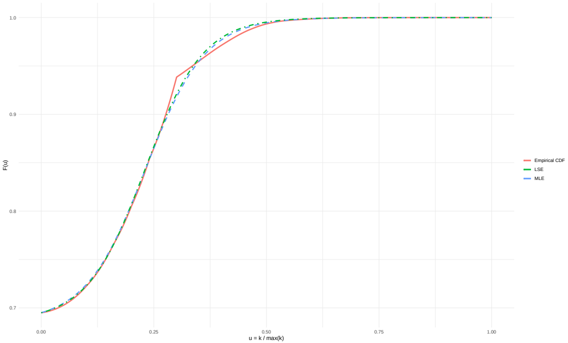
Structure $\mathcal{T}^{5,3}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-15.00	11.50	3.06×10^{-7}	99019.39
LSE	-15.56	11.94	1.75×10^{-7}	153537.76

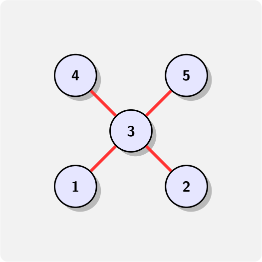
Table: Estimated parameters for $\mathcal{T}^{5,3}$ with $p = 0.1$

CDF vs MBBEFD Fit



Observations: The fit is accurate at the boundaries, with slight deviations in the middle of the range. This pattern is consistent across both estimation methods.

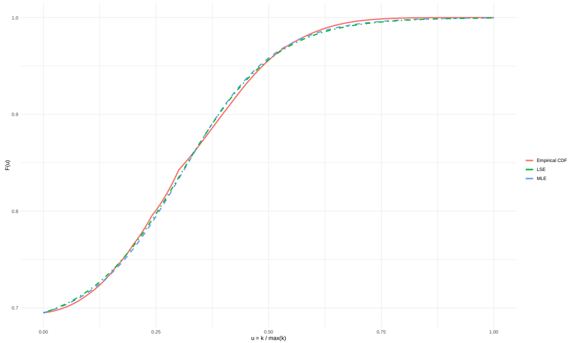
Structure $\mathcal{T}^{5,3}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-9.93	6.81	4.86×10^{-5}	911.39
LSE	-9.56	6.58	7.05×10^{-5}	722.51

Table: Estimated parameters for $\mathcal{T}^{5,3}$, $p = 0.3$

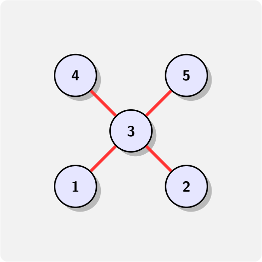
CDF vs MBBEFD Fit



Observations: The MBBEFD model provides a moderate fit to the CDF obtained from the model.

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,3}$, $p = 0.5$

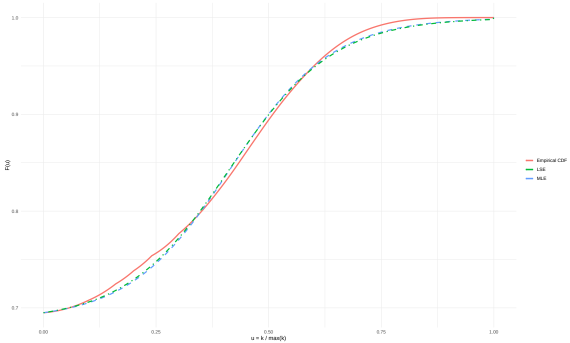
Structure $\mathcal{T}^{5,3}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-8.90	5.18	1.37×10^{-4}	179.11
LSE	-8.67	5.06	1.72×10^{-4}	158.47

Table: Estimated parameters for $\mathcal{T}^{5,3}$, $p = 0.5$

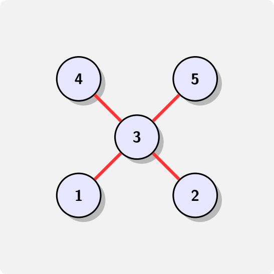
CDF vs MBBEFD Fit



Observations: The fit slightly oscillates around the analytical CDF, with alternating mild over- and underestimation. The effect is similar for both estimation methods.

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,3}$, $p = 0.7$

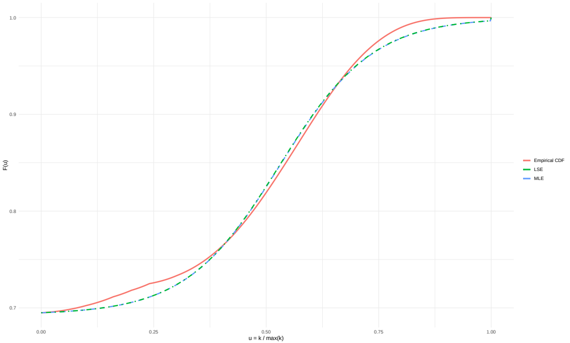
Structure $\mathcal{T}^{5,3}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-9.59	4.51	6.87×10^{-5}	91.49
LSE	-9.60	4.52	6.77×10^{-5}	92.74

Table: Estimated parameters for $\mathcal{T}^{5,3}$, $p = 0.7$

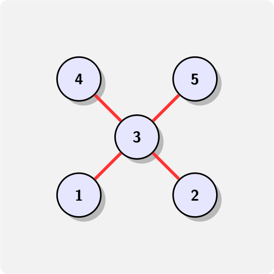
CDF vs MBBEFD Fit



Observations: Similar alternating bias as before, but with larger deviations. Both methods show the same pattern.

Fit of the MBBEFD model – Structure $\mathcal{T}^{5,3}$, $p = 0.9$

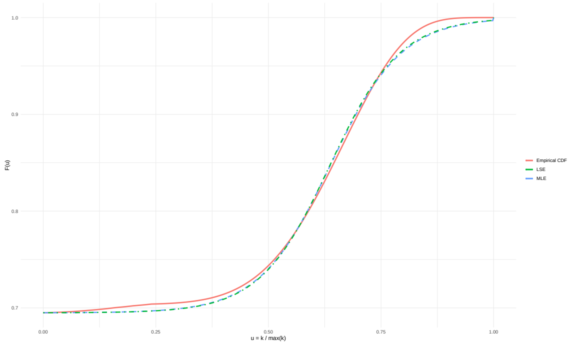
Structure $\mathcal{T}^{5,3}$



Method	$\hat{k}$	$\hat{\ell}$	$\hat{b}$	$\hat{g}$
MLE	-12.63	4.58	3.26×10^{-6}	98.55
LSE	-12.84	4.67	2.65×10^{-6}	107.25

Table: Estimated parameters for $\mathcal{T}^{5,3}$, $p = 0.9$

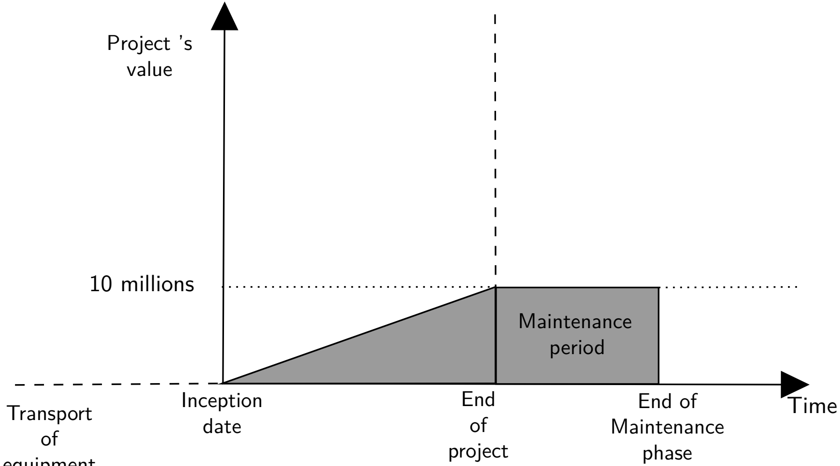
CDF vs MBBEFD Fit



Observations: Globally, the MBBEFD model provides a moderate fit to the CDF obtained from the model.

Pricing for construction insurance

Key Considerations



Adapted from [Parodi, 2023]

Pricing for construction insurance

Uniqueness of construction projects

- As highlighted in [Parodi, 2023], each construction project is **unique** and non-repetitive.
- This makes traditional **experience rating** methods less feasible.
- Usual actuarial practices (e.g., **rate change analysis**) are limited:
  - No renewal history;
  - Limited data at the individual client level.

Implication: Pricing must rely more on *project-specific characteristics* than on past client behavior.

Pricing for construction insurance

Non-standard time frames

- Construction policies typically cover **non-annual** periods.
- Risks exist throughout the project life cycle:
  - From groundworks to final touches;
  - Losses may occur at any stage.

Implication: Models must account for *irregular timing* of risks and exposures.

Pricing for construction insurance

Uneven risk exposure over time

- Premium earning pattern is **not uniform**.
- At the start:
  - Low exposure (e.g., empty lot).
- Over time:
  - Asset value and risk increase as construction progresses.

Implication: Pricing must reflect the *evolving risk profile* throughout the project.

Over the past decade, the construction industry has evolved to meet the growing needs for sustainability, energy efficiency, and aesthetics.

Mass timber stands out for its ecological benefits and exceptional structural properties, [Abdallah et al., 2015].

(a) Source: https://masstimberservices.com/products/clt/

(b) Source: [Levée et al., 2020]

(c) Source: http://clt-france.fr/

Mass timber construction and insurance challenges

However, mass timber construction projects face problems due to insurance ratemaking.

Research contributions in actuarial science:

- No research contributions on insurance ratemaking for mass timber in the actuarial literature.
- No research contributions on ratemaking of construction insurance in the actuarial literature.
- The ratemaking used by actuaries for construction insurance is not clearly known.

One crucial feature of actuarial pricing models is the Verisk/ISO construction classes:

Construction Class	Description	Construction Class	Description
1	Frame	2	Joisted masonry
3	Noncombustible	4	Masonry noncombustible
5	Modified fire resistive	6	Fire resistive

- Critic 1: The methodology to fix the classes is unclear.
- Critic 2: That approach seems to ignore the recent advances on construction methods and products.
- Critic 3: The methodology seems to ignore the US and Canada Building codes.
- In 2025-2026, Verisk aims to introduce a new class for mass timber products, [Shi and Kunzt, 2025].

Mass timber construction and insurance challenges

NRC.CANADA.CA

Figure 3. Photographs of mass timber test structure during construction.

A1-018329.1/A1-018487.1 4

Figure: Example of fire test - before. Source: Page 4 of [Su et al., 2023]

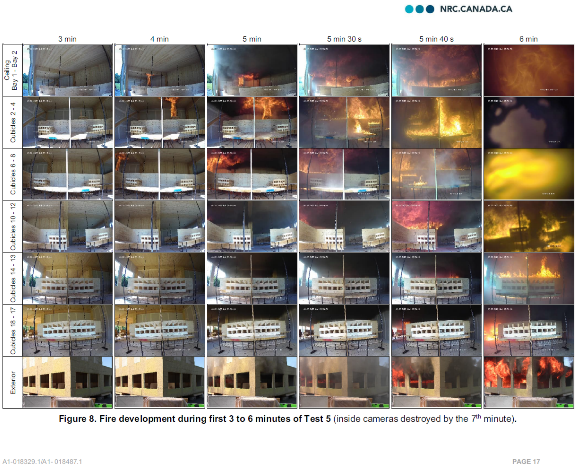


Figure: Example of fire test - after. Source: Page 17 of [Su et al., 2023]

Mass timber construction and insurance challenges

Insurers lack historical claims data which leads to conservative premiums.

This issue raises the pertinence of our actuarial fire-spreading model:

- The actuarial fire-spreading model aims to compensate for the lack of historical claim data.
- The results of fire tests can be integrated, such as [Zhang et al., 2015], [Brandon and Östman, 2016], [Su et al., 2018], [Brandon, 2018], [Brandon and Dagenais, 2018], and [Su et al., 2023].
- The actuarial fire-spreading model allows to include advances on construction methods and products.
- It serves as a platform for collaborations between P&C actuaries and civil/fire engineers.

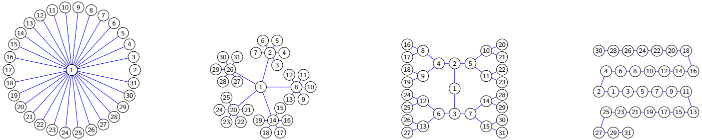
Additional comments:

- Key milestone: Inclusion of tall mass timber structures in the International Building Code in 2021.
- It allows for the construction of mass timber towers up to 18 stories in the Canada and U.S.
- Mass Timber Insurance Action Plan: link CSBA and CWC.
- Positive response of the insurance industry: link Zurich 2024 and Zurich 2021.
- Mass Timber: New Technology Drives a New Construction Class: link Verisk.
- Examples: link Top 25 Tallest Mass Timber Buildings in the World.

Conclusion

Other results:

- The methodology can be extended to larger trees with d nodes.
- Applicable to any configuration of nodes, offering flexibility in modeling complex structures.
- Applications related to farm insurance and fire contagion model in [Boucher et al., 2024].



Structure A   Structure B   Structure C   Structure D

Future works:

- Apply the actuarial fire-spreading model in the property exposure rating framework.
- Reinforce links between the research work from actuaries and civil/fire engineers.
- Investigate a dynamic version of the proposed actuarial fire-spreading model.

Conclusion

Thank you for your attention!
Asante!

I think every morning we wake up offers a new beginning, a new way of stepping into the day. I think every conversation could be a new chance. Yvonne Adhiambo Owuor

Acknowledgements

The research work was partially funded by

- NSERC Discovery grant program: JP Boucher, H Cossette, E Marceau
- CIRCERB, Université Laval, chair: Prof. Pierre Blanchet
- Chaire Co-Operators, UQaM, co-chairs: Prof. JP Boucher, M Pigeon
- Chaire d'actuariat, Université Laval

I think every morning we wake up offers a new beginning, a new way of stepping into the day. I think every conversation could be a new chance. Yvonne Adhiambo Owuor

References I

Abdallah, A., Boucher, J.-P., and Cossette, H. (2015).
Modeling dependence between loss triangles with hierarchical archimedean copulas.
ASTIN Bulletin: The Journal of the IAA, 45(3):577–599.

Albrecher, H., Beirlant, J., and Teugels, J. (2017).
Reinsurance.
Wiley.

Barrois, M. T. (1835).
Essai sur l'application du calcul des probabilités aux assurances contre l'incendie.
Danel.

Bernegger, S. (1997).
The Swiss Re exposure curves and the MBBEFD distribution class.
ASTIN Bulletin: The Journal of the IAA, 27(1):99–111.

References II

Boucher, J.-P., Crainic, A., LeBlanc, A., and Masse, V. (2024).
Modeling of fire contagion in farms insurance.
Variance, 17(1).

Brandon, D. (2018).
Fire safety challenges of tall wood buildings phase 2: Task 4 - engineering methods.
Technical Report Fire Protection Research Foundation.

Brandon, D. and Dagenais, C. (2018).
Fire safety challenges of tall wood buildings phase 2: Task 5 - experimental study of delamination of cross laminated timber (clt) in fire.
Technical Report Fire Protection Research Foundation.

References III

- Brandon, D. and Östman, B. (2016).
  Fire safety challenges of tall wood buildings phase 2: Task 1 - literature review.
  Technical Report Fire Protection Research Foundation.
- Cheng, H. and Hadjisophocleous, G. V. (2009).
  The modeling of fire spread in buildings by Bayesian network.
  Fire Safety Journal, 44(6):901–908.
- Gaskin, J. and Yung, D. (1993).
  Canadian and U.S.A. fire statistics for use in the risk-cost assessment model, IRC internal report no. 637.
  Technical report, National Research Council Canada.

References IV

Jiang, J., Wang, B., Yi, X., Wang, Q., Chen, W., and Ye, J. (2025).
Probability-based horizontal fire spread scenarios in framed structures and data-driven prediction model.
Journal of Building Engineering, 101:111798.

Kim, J., Dietz, J. E., and Matson, E. T. (2016).
Simulation modeling of a statistical fire spread to respond fire accident in buildings.
In *2016 IEEE Symposium on Technologies for Homeland Security (HST)*, pages 1–6. IEEE.

Koller, D. and Friedman, N. (2009).
Probabilistic Graphical Models: Principles and Techniques.
MIT press.

References V

Lauritzen, S. L. (1996).
Graphical Models, volume 17.
Clarendon Press.

Levée, V., Lavoie, K., and Bellavance, S. T. (2020).
Arbora : Une structure en bois apparente dans un projet résidentiel de grande envergure.
Cecobois.

Li, X., Zhang, X., and Hadjisophocleous, G. (2013).
Fire risk analysis of a 6-storey residential building using curisk.
Procedia engineering, 62:609–617.

Ling, W.-C. T. and Williamson, R. B. (1985).
Modeling of fire spread through probabilistic networks.
Fire safety journal, 9(3):287–300.

References VI

Mou, X. (2009).
Quantitative fire risk analysis case study using curisk, master thesis, master of applied science.
Technical report, Carleton University.

Parodi, P. (2023).
Pricing in General Insurance.
Chapman and Hall/CRC.

Parodi, P., Thrumble, D., Watson, P., Ji, Z., Wang, A., Bhatia, I., Lees, J., Mealy, S., Shah, R., Dharamshi, P., et al. (2024).
Loss modelling from first principles.
British Actuarial Journal, 29:e18.

References VII

Parodi, P. and Watson, P. (2019).
Property graphs–a statistical model for fire and explosion losses based on graph theory.
ASTIN Bulletin: The Journal of the IAA, 49(2):263–297.

Rychlik, I. and Rydén, J. (2006).
Probability and Risk Analysis.
Springer.

Shi, X. L. and Kunzt, K. (2025).
Mass timber: New technology drives a new construction class.
Technical report, Verisk.

References VIII

Su, J., Gibbs, E., Weinfurter, M., Lafrance, P.-S., Gratton, K., Frade, A., and Leroux, P. (2023).
Large-scale fire tests of a mass timber building structure for MTDFTP.
National Research Council of Canada.

Su, J., Lafrance, P.-S., Hoehler, M., and Bundy, M. (2018).
Fire safety challenges of tall wood buildings phase 2: Tasks 2 and 3 - cross laminated timber compartment fire tests.
Technical Report Fire Protection Research Foundation.

Wainwright, M. J., Jordan, M. I., et al. (2008).
Graphical models, exponential families, and variational inference.
Foundations and Trends® in Machine Learning, 1(1–2):1–305.

References IX

Wuthrich, M. V. (2023).
Non-life insurance: mathematics & statistics.
Available at SSRN 2319328.

Zhang, X., Mehaffey, J., and Hadjisophocleous, G. (2015).
Fire Risk Analysis Model CUrisk.
Department of Civil and Environmental Engineering, Carleton University.

Illustration of the impact of propagation probabilities (additional results)

For $p = 0.3$ and $(\Pr(I_s = 1|\Theta = s), s \in \mathcal{V}) = (0.2, 0.2, 0.4, 0.2, 0.1)$:

i	c_i	$E[Y_i]$	$E[I_i^{\mathcal{T}^{5,1}}]$	$E[D_i^{\mathcal{T}^{5,1}}]$	$E[I_i^{\mathcal{T}^{5,2}}]$	$E[D_i^{\mathcal{T}^{5,2}}]$	$E[I_i^{\mathcal{T}^{5,3}}]$	$E[D_i^{\mathcal{T}^{5,3}}]$
1	100	67.17	0.05	3.15	0.05	3.16	0.09	6.17
2	80	48.5	0.1	4.64	0.1	4.66	0.09	4.45
3	150	113	0.24	27.15	0.24	27.27	0.25	27.74
4	120	86.21	0.11	9.47	0.11	9.38	0.11	9.49
5	50	25.5	0.04	0.96	0.08	1.96	0.08	1.99

i	$C^{\mathrm{Var}}_{\mathcal{T}^{5,1}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,1}})}$	$C^{\mathrm{Var}}_{\mathcal{T}^{5,2}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,2}})}$	$C^{\mathrm{Var}}_{\mathcal{T}^{5,3}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,3}})}$
1	5.55	0.07	5.52	0.07	11.75	0.14
2	8.92	0.11	8.96	0.11	7.82	0.09
3	44.27	0.56	44.96	0.56	46.01	0.53
4	18.62	0.24	17.84	0.22	17.63	0.2
5	1.87	0.02	3.5	0.04	3.55	0.04

Illustration of the impact of propagation probabilities (additional results)

For $p = 0.1$ and $(\Pr(I_s = 1|\Theta = s), s \in \mathcal{V}) = (0.2, 0.2, 0.4, 0.2, 0.1)$:

i	c_i	$E[Y_i]$	$E[I_i^{\mathcal{T}^{5,1}}]$	$E[D_i^{\mathcal{T}^{5,1}}]$	$E[I_i^{\mathcal{T}^{5,2}}]$	$E[D_i^{\mathcal{T}^{5,2}}]$	$E[I_i^{\mathcal{T}^{5,3}}]$	$E[D_i^{\mathcal{T}^{5,3}}]$
1	100	67.17	0.02	1.63	0.02	1.63	0.04	2.86
2	80	48.5	0.04	2.15	0.04	2.16	0.04	2.07
3	150	113	0.23	25.57	0.23	25.62	0.23	25.82
4	120	86.21	0.06	5.41	0.06	5.37	0.06	5.38
5	50	25.5	0.01	0.29	0.03	0.7	0.03	0.71

i	$C^{\mathrm{Var}}_{\mathcal{T}^{5,1}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,1}})}$	$C^{\mathrm{Var}}_{\mathcal{T}^{5,2}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,2}})}$	$C^{\mathrm{Var}}_{\mathcal{T}^{5,3}}(D_i)$	% of $\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,3}})}$
1	1.81	0.03	1.78	0.03	5.16	0.08
2	3.62	0.06	3.61	0.06	3.18	0.05
3	43.68	0.73	43.98	0.73	44.31	0.7
4	10.1	0.17	9.75	0.16	9.45	0.15
5	0.38	0.01	1.24	0.02	1.23	0.02

Illustration of the impact of propagation probabilities (additional results)

For $p = 0.3$ and $(\Pr(I_s = 1|\Theta = s), s \in \mathcal{V}) = (0.2, 0.2, 0.4, 0.2, 0.1)$:

i	j	$\rho^{\mathcal{T}^{5,1}}_{I_i,I_j}$	$\mathrm{Cov}^{\mathcal{T}^{5,1}}(D_i, D_j)$	$\rho^{\mathcal{T}^{5,2}}_{I_i,I_j}$	$\mathrm{Cov}^{\mathcal{T}^{5,2}}(D_i, D_j)$	$\rho^{\mathcal{T}^{5,3}}_{I_i,I_j}$	$\mathrm{Cov}^{\mathcal{T}^{5,3}}(D_i, D_j)$
1	2	0.46	92.6	0.46	92.83	0.19	52.7
1	3	0.15	100.54	0.15	100.72	0.44	419.71
2	3	0.43	298.86	0.43	299.62	0.44	303.06
1	4	0.05	17.29	0.05	17.52	0.17	91.3
2	4	0.17	64.22	0.17	64.5	0.17	65.92
3	4	0.41	529.97	0.41	531.31	0.41	536.12
1	5	0.02	1.22	0.07	6.61	0.21	28.24
2	5	0.08	5.28	0.21	20.13	0.21	20.39
3	5	0.19	44.78	0.48	158.12	0.48	159.93
4	5	0.5	65.69	0.2	35.57	0.19	35.89

Illustration of the impact of propagation probabilities (additional results)

For $(\Pr(I_s = 1|\Theta = s), s \in \mathcal{V}) = (0.2, 0.2, 0.4, 0.2, 0.1)$:

p	$E[B^{\mathcal{T}^{5,1}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,1}})$	$E[B^{\mathcal{T}^{5,2}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,2}})$	$E[B^{\mathcal{T}^{5,3}}]$	$VaR_{0.9}(B^{\mathcal{T}^{5,3}})$
0.9	93	348	93.55	348	95.3	349
0.7	74.02	290	75.21	294	79.07	306
0.5	58.26	233	59.58	237	63.92	255
0.3	45.37	171	46.42	178	49.85	199
0.1	35.04	133	35.47	135	36.85	139

p	$\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,1}})}$	$\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,2}})}$	$\sqrt{\mathrm{Var}(B^{\mathcal{T}^{5,3}})}$
0.9	146.98	147.55	149.45
0.7	123.07	124.43	129.3
0.5	100.42	102.11	108.49
0.3	79.23	80.77	86.75
0.1	59.59	60.36	63.33

Illustration of the impact of tree structures

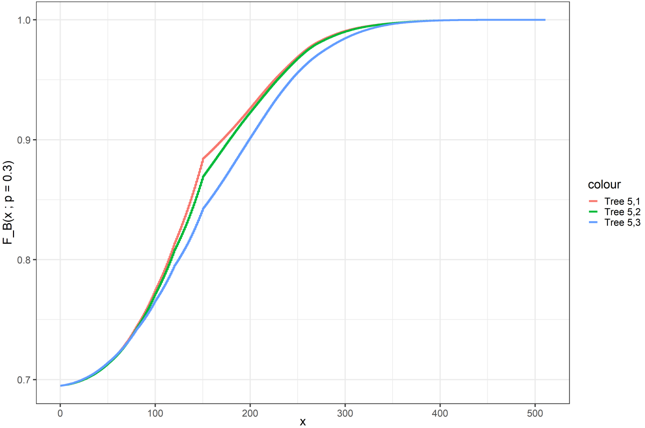


Figure: $F_{B^{\mathcal{T}}}(x)$ for trees $\mathcal{T}^{5,1}, \mathcal{T}^{5,2}, \mathcal{T}^{5,3}$ and $p = 0.3$

Abstract

We present an actuarial fire-spreading model based on tree-structured probabilistic graphical models with applications to insurance. We propose a general framework to model fire propagation, where each site is structured as a tree with units connected by edges. Our analysis focuses on two specific configurations: tree-structured sites with four and five units. For each configuration, we provide a stochastic representation of the total fire losses, and analyze the impact of propagation probabilities using stochastic orders. The model is further examined through a numerical application, offering insights into the practical implications for insurance, particularly in calculating premiums and risk measures. We apply the actuarial fire-spreading model to assess risks associated with construction projects involving mass timber, such as cross-laminated timber (CLT). We aim to propose a more detailed modeling of fire damage costs by integrating the results of laboratory studies on different construction materials and risk reduction measures (addition of fire doors, for example). This approach compensates the lack of data on innovative construction products and integrate results obtained by engineers about them.